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Buckling analysis of a laminated cylindrical shell under torsion subjected to mixed boundary conditions

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Abstract

A new efficient method is developed in this paper for buckling analysis of a cross-ply laminated cylindrical shell under torsion subjected to mixed boundary conditions. The transverse shear is taken into account by a first-order theory with a shear correction factor of 5/6. The mixed boundary conditions include conditions in forces as well as conditions in displacements, and these forces and displacements are selected as basic unknowns. The other displacements and forces are expressed in terms of the basic unknowns by taking inverse of a matrix composed of operators. The equations of buckled equilibrium in terms of the basic unknowns are solved with double trigonometric series which satisfy the mixed boundary conditions. Comparison of the obtained numerical results with those given in the literature based on completely clamped boundary conditions checks with the fact that the mixed boundary conditions yield appreciably lower buckling load and less circumferential wave number than the completely clamped boundary conditions. The curves in the figures show how the difference in buckling loads between the two kinds of boundary conditions varies when the length and thickness of the shell vary. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

The buckling of cylindrical shells under torsion (torsional buckling) has been much less studied in contrast to the buckling under axial compression (compressive buckling), especially for anisotropic shells. The main difficulty involved in the torsional buckling is in that firstly the nonlinear terms in the equations of buckled equilibrium include partial derivatives both in axial direction and in circumferential direction, so that the single-wave buckling mode, which is useful in the analysis of compressive buckling, does not work, and secondly the boundary conditions in torsional force T_{xy} and in torsional moment M_{xy} can hardly be satisfied beforehand by the assumed buckling mode. Simitses and Shaw (1985) and Hui and Du (1987) solve the difficulty for Donnell-type shallow

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shells without transverse shear by introducing a stress function and assuming a multi-wave buckling mode such as truncated trigonometric series in θ with the coefficients being functions of x, where x is the axial coordinate in length and θ is the circumferential coordinate in angle. Thus the equations of equilibrium and compatibility reduce to ordinary differential equations for the coefficients of the series, and the original boundary conditions yield boundary conditions for the solution of these ordinary differential equations.

In principle, the above solution procedure may apply to non-shallow shells with transverse shear. But in this case the number of the unknowns will be five and, due to the use of non-shallow shell theory, no simple stress function can be introduced to reduce that number. The above solution procedure will then be much more involved. To solve the problem, Tabiei and Simitses (1994) express the equations of equilibrium in terms of the three displacements and two rotations and solve them with double trigonometric series which vanish at the boundaries. Tabiei and Simitses' method is useful when all boundary conditions are in displacements and rotations, i.e., the shell is completely clamped at its ends, or when the shell is long enough so that the boundary conditions play little role. However, in the case of mixed boundary conditions, some of the conditions are given as boundary forces. They can hardly be satisfied beforehand by the double trigonometric series for displacements and rotations assumed by Tabiei and Simitses (1994). That is why the torsional buckling of transversely shear deformable shells under mixed boundary conditions has not yet been studied much.

In many practical cases, however, the real physical boundaries may be described better by mixed boundary conditions than by completely clamped boundary conditions. Therefore, solution methods are needed to deal with mixed boundary conditions. The present study undertakes this task to develop such a method. In the present study, the displacements (including rotations) and forces (including moments) appearing in the mixed boundary conditions are selected as basic unknowns and are expressed in terms of different kinds of double trigonometric series which satisfy the mixed boundary conditions. Then, by taking inverse of a matrix composed of operators, all other displacements and forces, and hence the five equations of buckled equilibrium, can be expressed in terms of the five basic unknowns. Finally, the application of the Galerkin procedure to the equations of buckled equilibrium leads to an eigenvalue problem. The numerical results of examples show that the mixed boundary conditions lead to lower buckling load and less circumferential wave number of the buckling mode than the completely clamped boundary conditions.

2. Basic equations

A thin cross-ply cylindrical shell subjected to torsion is under consideration. In the first-order shear-deformation theory the displacement field of a cylindrical shell is

$$u_x = u + z\psi, \quad u_y = v + z\varphi, \quad u_z = w \tag{1}$$

where (x, y, z) are, respectively, the axial, circumferential and normal coordinates in length, (u_x, u_y, u_z) are their corresponding displacement components at an arbitrary point of the shell, and (u, v, w) are those components at the middle surface. The linear strains at an arbitrary point of the shell can be derived from the displacement field,

$$\bar{\varepsilon}_x = \frac{\partial u}{\partial x} + z\frac{\partial \psi}{\partial x}, \quad \bar{\varepsilon}_y = \frac{\partial v}{\partial y} + \frac{w}{R} + z\frac{\partial \varphi}{\partial y}, \quad \bar{\gamma}_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x}\right)$$

$$\bar{\gamma}_{xz} = \frac{\partial w}{\partial x} + \psi, \quad \bar{\gamma}_{yz} = \frac{\partial w}{\partial y} - \frac{y}{R} + \varphi \tag{2}$$

These strains coincide with the general first-order shear-deformation kinematic relations of Tabiei and Simitses (1994) for linear case, if the thin shell assumption $z/R \ll 1$ is taken into account.

The definitions of the stress resultants are as usual

$$T_x = \int \sigma_x \,\mathrm{d}z, \quad T_y = \int \sigma_y \,\mathrm{d}z, \quad M_x = \int \sigma_x z \,\mathrm{d}z, \quad M_y = \int \sigma_y z \,\mathrm{d}z$$

$$T_{xy} = T_{yx} = \int \tau_{xy} \,\mathrm{d}z, \quad M_{xy} = M_{yx} = \int \tau_{xy} z \,\mathrm{d}z$$

$$Q_x = \int \tau_{xz} \,\mathrm{d}z, \quad Q_y = \int \tau_{yz} \,\mathrm{d}z \tag{3}$$

where

$$\int = \int_{-h/2}^{h/2}, \; h \text{ is the thickness of the shell}$$

By introducing vectors

$$\{\sigma\} = (T_x, T_y, T_{xy}, M_x, M_y, M_{xy}, Q_y, Q_x)^T$$

$$\{\varepsilon\} = (\varepsilon_x, \varepsilon_y, \gamma_{xy}, \kappa_x, \kappa_y, \kappa_{xy}, \gamma_{yz}, \gamma_{xz})^T$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\kappa_x = \frac{\partial \psi}{\partial x}, \quad \kappa_y = \frac{\partial \varphi}{\partial y}, \quad \kappa_{xy} = \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x}, \quad \gamma_{yz} = \bar{\gamma}_{yz}, \quad \gamma_{xz} = \bar{\gamma}_{xz}$$

are the strains and curvature changes of the middle surface, the constitutive equations for cross-ply laminates can be written in matrix form

$$\{\sigma\} = [C]\{\varepsilon\} \tag{4}$$

where

$$[C] = \begin{bmatrix} [A] & [B] & \\ [B] & [D] & \\ & & [A^*] \end{bmatrix}, \quad [A] = \begin{bmatrix} A_{11} & A_{12} & \\ A_{12} & A_{22} & \\ & & A_{66} \end{bmatrix}, \quad [B] = \begin{bmatrix} B_{11} & B_{12} & \\ B_{12} & B_{22} & \\ & & B_{66} \end{bmatrix}$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & \\ D_{12} & D_{22} & \\ & & D_{66} \end{bmatrix}, \quad [A^*] = \begin{bmatrix} A_{44} & \\ & A_{55} \end{bmatrix}$$

The equations of equilibrium developed by Stein (1986) are used in the present study. For a thin shell, after imposing the following three assumptions about the nonlinear terms:

(a) The only load is the boundary torques, so that all nonlinear terms not related to the torsional force T_{xy} are neglected;
(b) All nonlinear terms not related to w or its derivatives are neglected;
(c) All nonlinear terms not in the equation of normal equilibrium are neglected;

these equations reduce to

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_{xy}}{\partial y} = 0 \tag{5a}$$

$$\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{Q_y}{R} = 0 \tag{5b}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - \frac{T_y}{R} + 2T_{xy}\frac{\partial^2 w}{\partial x\,\partial y} = 0 \tag{5c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{5d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{5e}$$

and the variationally consistent boundary conditions can be written in the form

$$(T_x \delta u)|_0^L = 0 \tag{6a}$$

$$[(T_{xy} - S)\delta v]|_0^L = 0 \tag{6b}$$

$$\left[\left(T_{xy}\frac{\partial w}{\partial y} + Q_x\right)\delta w\right]\Big|_0^L = 0 \tag{6c}$$

$$(M_x \delta\psi)|_0^L = 0 \tag{6d}$$

$$(M_{xy}\delta\varphi)|_0^L = 0 \tag{6e}$$

$$(T_{xy}\delta u)|_0^{2\pi} = 0 \tag{7a}$$

$$(T_y \delta v)|_0^{2\pi} = 0 \tag{7b}$$

$$\left[\left(T_{xy}\frac{\partial w}{\partial x} + Q_y\right)\delta w\right]\Big|_0^{2\pi} = 0 \tag{7c}$$

$$(M_{xy}\delta\psi)|_0^{2\pi} = 0 \tag{7d}$$

$$(M_y\delta\varphi)|_0^{2\pi} = 0 \tag{7e}$$

The symbols $|_0^L$ and $|_0^{2\pi}$ are defined as

$$(T_x\delta u)|_0^L = (T_x\delta u)_{x=L} - (T_x\delta u)_{x=0}, \quad \text{and so on;}$$

$$(T_{xy}\delta u)|_0^{2\pi} = (T_{xy}\delta u)_{\theta=2\pi} - (T_{xy}\delta u)_{\theta=0}, \quad \text{and so on}$$

where L and R are, respectively, the length and radius of the shell, $\theta = y/R$ and S is the torsional force per unit length of y produced by the boundary torques. Conditions (7a–e) can be satisfied by the requirement that all displacements and forces appearing in these conditions be periodic in θ with a period equal to 2π. For conditions (6a–e), the following mixed boundary conditions are assumed

$$T_x = 0, \quad T_{xy} = S, \quad w = 0, \quad M_x = M_{xy} = 0 \quad \text{at } x = 0, L \tag{8}$$

It is obvious that conditions (6a–e) will be satisfied if conditions (8) are satisfied.

Equilibrium equations (5a–e) and boundary conditions (6a)–(7e) coincide with those of Tabiei and Simitses (1994) for thin shells using the first-order shear-deformation kinematic relations. The equations of Tabiei and Simitses (1994) is based on Donnell-type theory. This suggests that the error caused by the preceding assumptions (a)–(c) can be expected not to be larger than that of Donnell-type theory.

More complete equations of equilibrium for a thin cylindrical shell are given by Flugge (1960). These equations correspond to a system of kinematic relations in which all nonlinear terms of u, v and w are retained.

3. Solution

The five variables T_x, T_{xy}, w, M_x and M_{xy} appearing in the mixed boundary conditions (8) are selected as basic unknowns. The equations of buckled equilibrium (5a–e) can be transformed into a system of five equations in the basic unknowns by eliminating all other displacements and forces. To do this, the displacements and rotations u, v, ψ and φ are expressed in terms of the basic unknowns by using eqn (4) to obtain

$$[\Gamma]\begin{Bmatrix} u \\ v \\ \psi \\ \varphi \end{Bmatrix} = \begin{Bmatrix} T_x - A_{12}w/R \\ M_x - B_{12}w/R \\ T_{xy} \\ M_{xy} \end{Bmatrix} \tag{9}$$

where

$$[\Gamma] = \begin{bmatrix} A_{11}\frac{\partial}{\partial x} & A_{12}\frac{\partial}{\partial y} & B_{11}\frac{\partial}{\partial x} & B_{12}\frac{\partial}{\partial y} \\ B_{11}\frac{\partial}{\partial x} & B_{12}\frac{\partial}{\partial y} & D_{11}\frac{\delta}{\delta x} & D_{12}\frac{\partial}{\partial y} \\ A_{66}\frac{\partial}{\partial y} & A_{66}\frac{\partial}{\partial x} & B_{66}\frac{\partial}{\partial y} & B_{66}\frac{\partial}{\partial x} \\ B_{66}\frac{\partial}{\partial y} & B_{66}\frac{\partial}{\partial x} & D_{66}\frac{\partial}{\partial y} & D_{66}\frac{\partial}{\partial x} \end{bmatrix}$$

Let Γ_{ij} represent the elements of $[\Gamma]$, Λ_{ij} be the cofactor of the element Γ_{ji}, and Λ be the determinant of $[\Gamma]$. They are all operators. The inverse of $[\Gamma]$ can be written as

$$[\Gamma]^{-1} = \Lambda^{-1}[\Lambda_{ij}] \tag{10}$$

Substitution of eqn (10) into eqn (9) gives

$$\Lambda \begin{Bmatrix} u \\ v \\ \psi \\ \varphi \end{Bmatrix} = [\Lambda_{ij}] \begin{Bmatrix} T_x - A_{12}w/R \\ M_x - B_{12}w/R \\ T_{xy} \\ M_{xy} \end{Bmatrix} \tag{11}$$

The stress resultants T_y, M_y, Q_x and Q_y can then be expressed in terms of the basic unknowns through the constitutive equations:

$$T_y = A_{12}\frac{\partial u}{\partial x} + A_{22}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) + B_{12}\frac{\partial \psi}{\partial x} + B_{22}\frac{\partial \varphi}{\partial y} \tag{12a}$$

$$M_y = B_{12}\frac{\partial u}{\partial x} + B_{22}\left(\frac{\partial v}{\partial y} + \frac{w}{R}\right) + D_{12}\frac{\partial \psi}{\partial x} + D_{22}\frac{\partial \varphi}{\partial y} \tag{12b}$$

$$Q_x = A_{55}\left(\frac{\partial w}{\partial x} + \psi\right) \tag{12c}$$

$$Q_y = A_{44}\left(\frac{\partial w}{\partial y} - \frac{v}{R} + \varphi\right) \tag{12d}$$

By application of the operator Λ to each side of eqns (5b–e) and using eqns (12a–d) and eqn (11) to eliminate T_y, M_y, Q_x and Q_y, eqns (5a–e) can be transformed into a system of five final equations in the five basic unknowns. These final equations are too lengthy to be presented here.

The five final equations are solved with truncated double trigonometric series

$$T_x = \Sigma SF2 \sin \alpha_m x(T_{mn}\cos n\theta + \bar{T}_{mn}\sin n\theta) \tag{13a}$$

$$T_{xy} = S + \Sigma(\cos \alpha_{m-1}x - \cos \alpha_{m+1}x)(S_{mn}\cos n\theta + \bar{S}_{mn}\sin n\theta) \tag{13b}$$

$$w = \Sigma \sin \alpha_m x(W_{mn}\cos n\theta + \bar{W}_{mn}\sin n\theta) \tag{13c}$$

$$M_x = \Sigma \sin \alpha_m x (H_{mn} \cos n\theta + \bar{H}_{mn} \sin n\theta) \quad (13d)$$

$$M_{xy} = \Sigma(\cos \alpha_{m-1} x - \cos \alpha_{m+1} x)(F_{mn} \cos n\theta + \bar{F}_{mn} \sin n\theta) \quad (13e)$$

where, and in the following, the symbols

$$\Sigma = \sum_{m=1}^{M} \sum_{n=1}^{M}, \quad \alpha_m = \frac{m\pi}{L}$$

are used for brevity in writing. The constant coefficients T_{mn}, $\bar{T}_{mn}, \ldots, \bar{F}_{mn}$ are to be determined. Trigonometric series (13) satisfy the mixed boundary conditions (8).

Before using the Galerkin procedure, a closer inspection of the operators in eqns (11) is useful. The operator Λ is a fourth-order homogeneous polynomial in $\partial/\partial x$ and $\partial/\partial y$. Each term of the polynomial includes one of the operators $\partial^4/\partial x^4$, $\partial^4/(\partial x^2\, \partial y^2)$ and $\partial^4/\partial y^4$. Therefore, when the operator Λ is applied to a product of trigonometric functions such as $\sin \alpha_m x \cos n\theta$, a very simple result can be obtained

$$\Lambda(\sin \alpha_m x \cos n\theta) = \langle \Lambda \rangle_{mn} \sin \alpha_m x \cos n\theta \quad (14)$$

where the symbol $\langle \Lambda \rangle_{mn}$ is a number denoting the determinant of a matrix obtained by replacing $\partial/\partial x$ and $\partial/\partial y$ with α_m and β_n, respectively, in the operator Λ, where $\beta_n = n/R$. Each of the operators Λ_{ij} $(i,j = 1,2,3,4)$ is a third-order polynomial in $\partial/\partial x$ and $\partial/\partial y$. Each term of the polynomial includes one of the operators $\partial^3/\partial x^3$, $\partial^3/(\partial x^2\, \partial y)$, $\partial^3/(\partial x\, \partial y^2)$ and $\partial^3/\partial y^3$. Further, it is found that for an operator Λ_{ij}, the terms of its polynomial are either all in $\partial^3/\partial x^3$ and $\partial^3/(\partial x\, \partial y^2)$, or all in $\partial^3/\partial y$ and $\partial^3/(\partial x^2\, \partial y)$. Therefore, the operators Λ_{ij} are divided into two groups, one with odd power of $\partial/\partial x$ for all terms, the other with odd power of $\partial/\partial y$ for all terms. It is obvious that for an Λ_{ij} from the first group, the following relationships are valid:

$$\Lambda_{ij}(\sin \alpha_m x \cos n\theta) = -\langle \Lambda_{ij} \rangle_{mn} \cos \alpha_m x \cos n\theta \quad (15a)$$

$$\Lambda_{ij}(\sin \alpha_m x \sin n\theta) = -\langle \Lambda_{ij} \rangle_{mn} \cos \alpha_m x \sin n\theta \quad (15b)$$

$$\Lambda_{ij}(\cos \alpha_m x \cos n\theta) = \langle \Lambda_{ij} \rangle_{mn} \sin \alpha_m x \cos n\theta \quad (15c)$$

$$\Lambda_{ij}(\cos \alpha_m x \sin n\theta) = \langle \Lambda_{ij} \rangle_{mn} \sin \alpha_m x \sin n\theta \quad (15d)$$

and for an Λ_{ij} from the second group, eqns (15a–d) should be replaced by

$$\Lambda_{ij}(\sin \alpha_m x \cos n\theta) = \langle \Lambda_{ij} \rangle_{mn} \sin \alpha_m x \sin n\theta \quad (16a)$$

$$\Lambda_{ij}(\sin \alpha_m x \sin n\theta) = -\langle \Lambda_{ij} \rangle_{mn} \sin \alpha_m x \cos n\theta \quad (16b)$$

$$\Lambda_{ij}(\cos \alpha_m x \cos n\theta) = \langle \Lambda_{ij} \rangle_{mn} \cos \alpha_m x \sin n\theta \quad (16c)$$

$$\Lambda_{ij}(\cos \alpha_m x \sin n\theta) = -\langle \Lambda_{ij} \rangle_{mn} \cos \alpha_m x \cos n\theta \quad (16d)$$

where the symbol $\langle \Lambda_{ij} \rangle_{mn}$ is a number obtained by replacing $\partial/\partial x$ and $\partial/\partial y$ with α_m and β_n, respectively, in the operator Λ_{ij}.

Substituting the trigonometric series (13a–e) into the five final equations (not presented), taking into account eqns (15a–d) and (16a–d) and executing the Galerkin procedure yield the following algebraic equations:

$$f_{ij}^{ut}T_{ij}+f_{ij}^{uh}H_{ij}+\Sigma(\bar{f}_{ijmn}^{us}\bar{S}_{mn}+\bar{f}_{ijmn}^{uf}\bar{F}_{mn})=f_{ij}^{uw}W_{ij} \tag{17}$$

$$\bar{f}_{ij}^{ut}\bar{T}_{ij}+\bar{f}_{ij}^{uh}\bar{H}_{ij}+\Sigma(f_{ijmn}^{us}S_{mn}+f_{ijmn}^{uf}F_{mn})=\bar{f}_{ij}^{uw}\bar{W}_{ij} \tag{18}$$

$$f_{ij}^{vt}T_{ij}+f_{ij}^{vh}H_{ij}+\Sigma(\bar{f}_{ijmn}^{vs}\bar{S}_{mn}+\bar{f}_{ijmn}^{vf}\bar{F}_{mn})=f_{ij}^{vw}W_{ij} \tag{19}$$

$$\bar{f}_{ij}^{vt}\bar{T}_{ij}+\bar{f}_{ij}^{vh}\bar{H}_{ij}+\Sigma(f_{ijmn}^{vs}S_{mn}+f_{ijmn}^{vf}F_{mn})=\bar{f}_{ij}^{vw}\bar{W}_{ij} \tag{20}$$

$$f_{ij}^{wt}T_{ij}+f_{ij}^{wh}H_{ij}+\Sigma(\bar{f}_{ijmn}^{ws}\bar{S}_{mn}+\bar{f}_{ijmn}^{wf}\bar{F}_{mn})+f_{ij}^{ww}W_{ij}+2S\Sigma\alpha_m\beta_n\langle\Lambda\rangle_{mn}\eta_{im}\delta_{jn}\bar{W}_{mn}=0 \tag{21}$$

$$\bar{f}_{ij}^{wt}\bar{T}_{ij}+\bar{f}_{ij}^{wh}\bar{H}_{ij}+\Sigma(f_{ijmn}^{ws}S_{mn}+f_{ijmn}^{wf}F_{mn})+\bar{f}_{ij}^{ww}\bar{W}_{ij}-2S\Sigma\alpha_n\beta_n\langle\Lambda\rangle_{mn}\eta_{im}\delta_{jn}W_{mn}=0 \tag{22}$$

$$f_{ij}^{\psi t}T_{ij}+f_{ij}^{\psi h}H_{ij}+\Sigma(\bar{f}_{ijmn}^{\psi s}\bar{S}_{mn}+\bar{f}_{ijmn}^{wf}\bar{F}_{mn})=f_{ij}^{\psi w}W_{ij} \tag{23}$$

$$\bar{f}_{ij}^{\psi t}\bar{T}_{ij}+\bar{f}_{ij}^{\psi h}\bar{H}_{ij}+\Sigma(f_{ijmn}^{\psi s}S_{mn}+f_{ijmn}^{\psi f}F_{mn})=\bar{f}_{ij}^{\psi w}\bar{W}_{ij} \tag{24}$$

$$f_{ij}^{\phi t}T_{ij}+f_{ij}^{\phi h}H_{ij}+\Sigma(\bar{f}_{ijmn}^{\phi s}\bar{S}_{mn}+\bar{f}_{ijmn}^{\phi f}\bar{F}_{mn})=f_{ij}^{\phi w}W_{ij} \tag{25}$$

$$\bar{f}_{ij}^{\phi t}\bar{T}_{ij}+\bar{f}_{ij}^{\phi h}\bar{H}_{ij}+\Sigma(f_{ijmn}^{\phi s}S_{mn}+f_{ijmn}^{\phi f}F_{mn})=\bar{f}_{ij}^{\phi w}\bar{W}_{ij} \tag{26}$$

where $i = 1, 2, \ldots, M; j = 1, 2, \ldots, N$, and

$$\eta_{ij}=\frac{2[1-(-1)^{i+j}]i}{(i^2-j^2)\pi}, \quad \delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$$

In the derivation of eqns (21) and (22) the assumption

$$2T_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\approx 2S\frac{\partial^2 w}{\partial x\,\partial y}$$

has been made for linearization so that an eigenvalue problem can be obtained. All the coefficients in eqns (17)–(26) are listed in the Appendix.

From eqns (17) and (23), T_{ij} and H_{ij} can be solved in terms of $\bar{S}_{ij}$, $\bar{F}_{ij}$ and W_{ij}. Similarly, from eqns (18) and (24), $\bar{T}_{ij}$ and $\bar{H}_{ij}$ can be solved in terms of S_{ij}, F_{ij} and $\bar{W}_{ij}$. By using these results, T_{ij}, H_{ij}, $\bar{T}_{ij}$ and $\bar{H}_{ij}$ in eqns (21), (22), (20), (26), (19) and (25) are eliminated to give

$$\Sigma(\bar{C}_{ijmn}^{ws}\bar{S}_{mn}+\bar{C}_{ijmn}^{wf}\bar{F}_{mn})+C_{ij}^{ww}W_{ij}+2S\Sigma\alpha_m\beta_n\langle\Lambda\rangle_{mn}\eta_{im}\delta_{jn}\bar{W}_{mn}=0 \tag{27a}$$

$$\Sigma(C_{ijmn}^{ws}S_{mn}+C_{ijmn}^{wf}F_{mn})+\bar{C}_{if}^{ww}\bar{W}_{ij}-2S\Sigma\alpha_m\beta_n\langle\Lambda\rangle_{mn}\eta_{im}\delta_{jn}W_{mn}=0 \tag{27b}$$

$$\Sigma\begin{bmatrix}b_{ijmn}^{vs} & b_{ijmn}^{vf}\\ b_{ijmn}^{\phi s} & b_{ijmn}^{\phi f}\end{bmatrix}\begin{Bmatrix}S_{mn}\\ F_{mn}\end{Bmatrix}=\begin{Bmatrix}\bar{b}_{ij}^{vw}\\ \bar{b}_{ij}^{\phi w}\end{Bmatrix}\bar{W}_{ij} \tag{27c}$$

$$\Sigma\begin{bmatrix}\bar{b}_{ijmn}^{vs} & \bar{b}_{ijmn}^{vf}\\ \bar{b}_{ijmn}^{\phi s} & \bar{b}_{ijmn}^{\phi f}\end{bmatrix}\begin{Bmatrix}\bar{S}_{mn}\\ \bar{F}_{mn}\end{Bmatrix}=\begin{Bmatrix}b_{ij}^{vw}\\ b_{ij}^{\phi w}\end{Bmatrix}W_{ij} \tag{27d}$$

The expressions of the new coefficients appearing in eqns (27a–d) are not given here for the sake of brevity.

By introducing vectors

$$\{q\}=(S_{11},S_{21},\ldots,S_{M1},\ldots,S_{1N},S_{2N},\ldots,S_{MN},F_{11},F_{21},\ldots,F_{M1},\ldots,F_{1N},F_{2N},\ldots,F_{MN})^{\mathrm{T}}$$

$\{\bar{q}\}$ = the same as above but with a super bar on each element.

$\{\eta\} = (W_{11}, W_{21}, \ldots, W_{M1}, \ldots, W_{1N}, W_{2N}, \ldots, W_{MN})^{\mathrm{T}}$

$\{\bar{\eta}\}$ = the same as above but with a super bar on each element.

eqns (27a–d) can be transformed into matrix form

$$[\bar{\Omega}]\{\bar{q}\}+[\Delta]\{\eta\}+S[\bar{E}]\{\bar{\eta}\} = 0 \tag{28a}$$

$$[\Omega]\{q\}+[\bar{\Delta}]\{\bar{\eta}\}-S[E]\{\eta\} = 0 \tag{28b}$$

$$[\Phi]\{q\} = [\bar{\Psi}]\{\bar{\eta}\} \tag{28c}$$

$$[\bar{\Phi}]\{\bar{q}\} = [\Psi]\{\eta\} \tag{28d}$$

Again the expressions of the coefficient matrices in (28a–d) are not presented. Eliminating $\{q\}$ and $\{\bar{q}\}$ in eqns (28a–d) yields a set of equations in $\{\eta\}$ and $\{\bar{\eta}\}$,

$$[K]\{\eta\}+S[\bar{E}]\{\bar{\eta}\} = 0 \tag{29a}$$

$$[\bar{K}]\{\bar{\eta}\}-S[E]\{\eta\} = 0 \tag{29b}$$

where

$$[K] = [\bar{\Omega}]\,[\bar{\Phi}]^{-1}[\Psi]+[\Delta]$$

$$[\bar{K}] = [\Omega]\,[\Phi]^{-1}[\bar{\Psi}]+[\bar{\Delta}]$$

Eqns (29b) and (29a) give

$$\{\eta\} = \frac{1}{S}[E]^{-1}[\bar{K}]\{\bar{\eta}\} \tag{30}$$

$$([A]-S^2[I])\{\bar{\eta}\} = 0 \tag{31}$$

where $[I]$ is a unit matrix and $[A]$ is defined by

$$[A] = -[\bar{E}]^{-1}[K]\,[E]^{-1}[\bar{K}]$$

Eqn (31) is the eigenvalue problem for the buckling load, S_{cr}, and half of the buckling mode vector, $\{\bar{\eta}_{\mathrm{cr}}\}$. The other half, $\{\eta_{\mathrm{cr}}\}$, can be obtained with eqn (30). From eqn (31) it is found that if S_{cr} is a buckling load, then $-S_{\mathrm{cr}}$ is a buckling load, too. This conclusion is in agreement with the fact that if a boundary torque causes buckling, then the reverse of it causes buckling, too.

4. Numerical examples

The first example is a laminated cylindrical shell studied by Tabiei and Simitses (1994), with the ply properties

$$E_{11} = 149.619\,\mathrm{GPa},\quad E_{22} = E_{33} = 9.928\,\mathrm{GPa},\quad G_{12} = G_{13} = 4.481\,\mathrm{GPa}$$

$$G_{23} = 2.551\,\mathrm{GPa},\quad v_{12} = v_{13} = 0.28,\quad v_{23} = 0.45$$

and the stacking sequence $(0^\circ/90^\circ/0^\circ)_s$. It is obvious that the more terms are used in the truncated series (13a–e), the better results can be obtained and larger amount of computation is needed. The study of convergence indicates that $M = 6$ and $N = 10$ can guarantee that the buckling load to be obtained has more than four significant digits for all examples studied in this Section. Let S^*_{cr} and S_{cr} denote, respectively, the buckling loads based on the first-order shear deformation theory with and without a shear correction factor of 5/6. For $R/h = 100$, $L/R = 1$ and $h = 0.001905$ m, the calculation gives the buckling loads (in 10^6 N/m) $S^*_{cr} = 0.1391(8)$ and $S_{cr} = 0.1393(8)$, where the number in parentheses is the circumferential wave number. By comparison with $S_{cr} = 0.1568(9)$ given by Tabiei and Simitses (1994) based on the completely clamped boundary conditions, it can be seen that the mixed boundary conditions yield appreciably lower buckling load and less circumferential wave number than the completely clamped boundary conditions. However, for long shells, the difference is negligible. For instance, for $R/h = 100$ and $L/R = 5$, the present theory gives buckling loads $S^*_{cr} = S_{cr} = 0.0751(5)$ which is almost equal to $S_{cr} = 0.0757(6)$ given by Tabiei and Simitses (1994). The full profile of S_{cr} vs L/R is shown in Fig. 1.

It may be interesting to take a look at the buckling mode. First it is found that the buckling mode actually has a single wavelength in the circumferential direction while it is a combination of many waves of different wavelengths in the axial direction. For instance, for $R/h = 100$ and $L/R = 1$, the buckling mode is

$$w_{cr} = (-52.93 \sin \alpha_1 x + 73.11 \sin \alpha_2 x + 5.026 \sin \alpha_3 x + \cdots) \cos 5\theta + (15.71 \sin \alpha_1 x + 246.4 \sin \alpha_2 x - 1.492 \sin \alpha_3 x + \cdots) \sin 5\theta$$

in which all terms with the circumferential wave number $n \neq 5$ disappear because they are exactly zero. This finding suggests an alternative way to calculate the buckling load. By fixing the number n in series (13a–e); instead of taking summation for it, the procedure developed in the preceding sections becomes simpler. For each specified n, the simplified procedure gives a buckling load S_n.

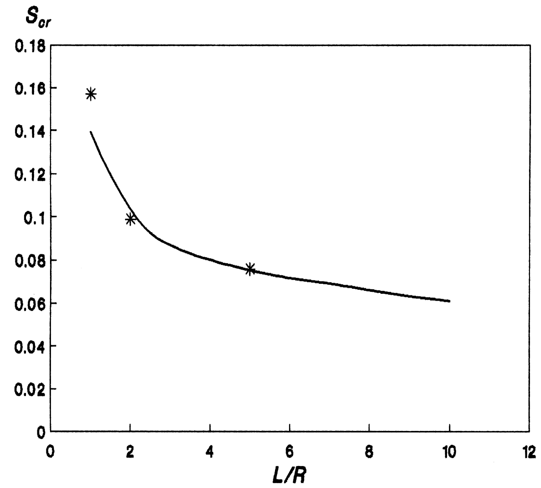

Fig. 1. The curve of S_{cr} vs L/R at $R/h = 100$ for Example 1 is based on mixed boundary conditions. The stars are results from Tabiei and Simitses (1994) based on completely clamped boundary conditions.

As usual, the lowest S_n for all n is the actual buckling load (critical load) S_{cr}, and the number of n which yields S_{cr} is the actual circumferential wave number of the buckling mode. But here in this paper the authors prefer presenting the theory with double trigonometric series, instead of single trigonometric series obtained by fixing n, because in the form of double trigonometric series the theory can be extended to cover more general cases such as postbuckling.

The second detail about the buckling mode worth mentioning is that for each buckling load, the computer gives two different buckling modes. This means that the corresponding eigenvalue is a twofold repeated root. As a matter of fact, the two buckling modes can be obtained from each other by suitable shift of the origin of the coordinate θ, which does not affect the buckling load. To show this, let

$$w_{c1} = (a_1 \sin\alpha_1 x + a_2 \sin\alpha_2 x + \cdots)\cos p\theta + (\bar{a}_1 \sin\alpha_1 x + \bar{a}_2 \sin\alpha_2 x + \cdots)\sin p\theta$$

be a buckling mode. The shift of the origin of θ by ξ/p makes $\cos p\theta$ and $\sin p\theta$ be replaced by $\cos(p\theta - \xi)$ and $\sin(p\theta - \xi)$, respectively, and w_{c1} be transformed to another mode

$$w_{c2} = [(a_1\cos\xi - \bar{a}_1 \sin\xi)\sin\alpha_1 x + (a_2 \cos\xi - \bar{a}_2\sin\xi)\sin\alpha_2 x + \cdots]\cos p\theta$$
$$+ [(a_1 \sin\xi + \bar{a}_1\cos\xi)\sin\alpha_1 x + (a_2\sin\xi + \bar{a}_2\cos\xi)\sin\alpha_2 x + \cdots]\sin p\theta$$

Thus, infinitely many such buckling modes can be obtained by assigning different values to ξ. But the number of linearly independent modes is at most two. Therefore, for each buckling load the computer gives only two different modes, not more, which represent the same actual physical mode.

Another example is a clamped unsymmetrically laminated shell cited from Hui and Du (1987). The stacking sequence is (90° in/0° out), with the ply properties

$$E_{11}/E_{22} = 10.0, \quad G_{12}/E_{22} = 0.5, \quad v_{12} = 0.25, \quad v_{23} = 0.4, \quad G_{23}/E_{22} = 0.27$$

For $R/h = 100$ and $L/R = 1$ the present theory gives the dimensionless critical shear stress

$$\tau_{cr} = \frac{S_{cr}R}{E_{22}T^2} = 0.4996(9)$$

while Hui and Du (1987) gives $\tau_{cr} = 0.5027(9.4)$. The two results are quite close in both buckling load and wave number. But for a shorter shell, say $L/R = 0.5$, the two results

$\tau_{cr} = 0.8017(10)$ from the present theory

$\tau_{cr} = 0.9713(10)$ from Hui and Du (1987)

are appreciably different. Comparison for various R/h is shown in Fig. 2.

5. Conclusions

A new efficient method is developed in this paper to deal with mixed boundary conditions for transversely shear deformable cylindrical shells. The main idea is to take the displacements and forces related to the boundary conditions as basic unknowns and to express those not related to the boundary conditions in terms of the basic unknowns by taking inverse of a matrix composed

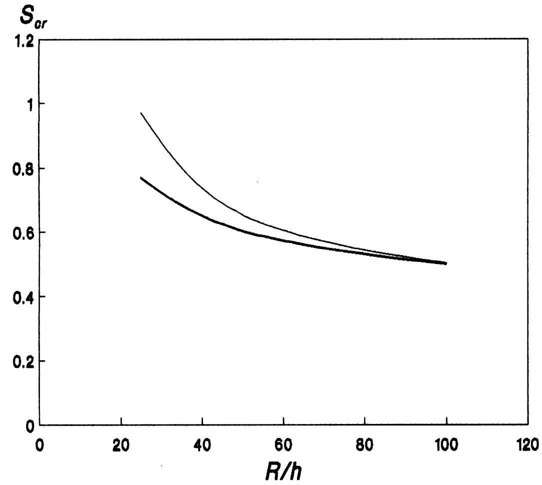

Fig. 2. The relationship between S_{cr} and R/h at $L/R = 1$ for Example 2. The thick curve is based on mixed boundary conditions. The thin curve is based on completely clamped boundary conditions from Hui and Du (1987).

of operators. The numerical results of two torsional buckling problems of such shells show that for short shells the present method makes improvement in calculation of buckling loads and wave numbers compared with the methods in the literature which are based on fully clamped boundaries.

Appendix

The coefficients in eqns (17)–(26) are listed below:

$$f^{ut}_{ij} = \alpha_i, \quad f^{uh}_{ij} = 0, \quad \bar{f}^{us}_{ijmn} = \beta_n(\delta_{i,m-1} - \delta_{i,m+1})\delta_{jn}, \quad \bar{f}^{uf}_{ijmn} = 0, \quad f^{uw}_{ij} = 0$$

$$\bar{f}^{ut}_{ij} = f^{ut}_{ij}, \quad \bar{f}^{uh}_{ij} = f^{uh}_{ij}, \quad f^{us}_{ijmn} = -\bar{f}^{us}_{ijmn}, \quad f^{uf}_{ijmn} = -\bar{f}^{uf}_{ijmn}, \quad \bar{f}^{uw}_{ij} = f^{uw}_{ij}$$

$$\bar{f}^{vt}_{ij} = A_{12}\alpha_i\beta_j\langle\Lambda_{11}\rangle_{ij} + (A_{22}\beta_j^2 + A_{44}/R^2)\langle\Lambda_{21}\rangle_{ij}$$
$$+ B_{12}\alpha_i\beta_j\langle\Lambda_{31}\rangle_{ij} + (B_{22}\beta_j^2 - A_{44}/R)\langle\Lambda_{41}\rangle_{ij}$$

$$\bar{f}^{vh}_{ij} = A_{12}\alpha_i\beta_j\langle\Lambda_{12}\rangle_{ij} + (A_{22}\beta_j^2 + A_{44}/R^2)\langle\Lambda_{22}\rangle_{ij}$$
$$+ B_{12}\alpha_i\beta_j\langle\Lambda_{32}\rangle_{ij} + (B_{22}\beta_j^2 - A_{44}/R)\langle\Lambda_{42}\rangle_{ij}$$

$$\begin{aligned} f^{vs}_{ijmn} = & -(\alpha_{m-1}\langle\Lambda\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & - A_{12}\beta_n(\alpha_{m-1}\langle\Lambda_{13}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{13}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & - (A_{22}\beta_n^2 + A_{44}/R^2)(\langle\Lambda_{23}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{23}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & - B_{12}\beta_n(\alpha_{m-1}\langle\Lambda_{33}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{33}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & - (B_{22}\beta_n^2 - A_{44}/R)(\langle\Lambda_{43}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{43}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \end{aligned}$$

$$f_{ijmn}^{vf} = -A_{12}\beta_n(\alpha_{m-1}\langle\Lambda_{14}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{14}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-(A_{22}\beta_n^2 + A_{44}/R^2)(\langle\Lambda_{24}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{24}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-B_{12}\beta_n(\alpha_{m-1}\langle\Lambda_{34}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{34}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-(B_{22}\beta_n^2 - A_{44}/R)(\langle\Lambda_{44}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{44}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$\bar{f}_{ij}^{vw} = -\frac{A_{22}+A_{44}}{R}\beta_j\langle\Lambda\rangle_{ij} + \frac{A_{12}}{R}\bar{f}_{ij}^{vt} + \frac{B_{12}}{R}\bar{f}_{ij}^{vh}$$

$$f_{ij}^{vt} = -\bar{f}_{ij}^{vt}, \quad f_{ij}^{vh} = -\bar{f}_{ij}^{vh}, \quad \bar{f}_{ijmn}^{vs} = f_{ijmn}^{vs}, \quad \bar{f}_{ijmn}^{vf} = f_{ijmn}^{vf}, \quad f_{ij}^{vw} = -\bar{f}_{ij}^{vw}$$

$$f_{ij}^{wt} = -\frac{A_{12}}{R}\alpha_i\langle\Lambda_{11}\rangle_{ij} - \frac{A_{22}+A_{44}}{R}\beta_j\langle\Lambda_{21}\rangle_{ij}$$
$$-\left(\frac{B_{12}}{R} - A_{55}\right)\alpha_i\langle\Lambda_{31}\rangle_{ij} - \left(\frac{B_{22}}{R} - A_{44}\right)\beta_j\langle\Lambda_{41}\rangle_{ij}$$

$$f_{ij}^{wh} = -\frac{A_{12}}{R}\alpha_i\langle\Lambda_{12}\rangle_{ij} - \frac{A_{22}+A_{44}}{R}\beta_j\langle\Lambda_{22}\rangle_{ij}$$
$$-\left(\frac{B_{12}}{R} - A_{55}\right)\alpha_i\langle\Lambda_{32}\rangle_{ij} - \left(\frac{B_{22}}{R} - A_{44}\right)\beta_j\langle\Lambda_{42}\rangle_{ij}$$

$$\bar{f}_{ijmn}^{ws} = -\frac{A_{12}}{R}(\alpha_{m-1}\langle\Lambda_{13}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{13}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-\frac{A_{22}+A_{44}}{R}\beta_n(\langle\Lambda_{23}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{23}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-\left(\frac{B_{12}}{R} - A_{55}\right)(\alpha_{m-1}\langle\Lambda_{33}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{33}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-\left(\frac{B_{22}}{R} - A_{44}\right)\beta_n(\langle\Lambda_{43}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{43}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$\bar{f}_{ijmn}^{wf} = -\frac{A_{12}}{R}(\alpha_{m-1}\langle\Lambda_{14}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{14}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-\frac{A_{22}+A_{44}}{R}\beta_n(\langle\Lambda_{24}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{24}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-\left(\frac{B_{12}}{R} - A_{55}\right)(\alpha_{m-1}\langle\Lambda_{34}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{34}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$
$$-\left(\frac{B_{22}}{R} - A_{44}\right)\beta_n(\langle\Lambda_{44}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{44}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$f_{ij}^{ww} = -\left(A_{55}\alpha_i^2 + A_{44}\beta_j^2 + \frac{A_{22}}{R^2}\right)\langle\Lambda\rangle_{ij} - \frac{A_{12}}{R}f_{ij}^{wt} - \frac{B_{12}}{R}f_{ij}^{wh}$$

$$\bar{f}_{ij}^{wt} = f_{ij}^{wt},\quad \bar{f}_{ij}^{wh} = f_{ij}^{wh},\quad f_{ijmn}^{ws} = -\bar{f}_{ijmn}^{ws},\quad f_{ijmn}^{wf} = -\bar{f}_{ijmn}^{wf},\quad \bar{f}_{ij}^{ww} = f_{ij}^{ww}$$

$$f_{ij}^{\psi t} = A_{55}\langle\Lambda_{31}\rangle_{ij}$$

$$f_{ij}^{\psi h} = \alpha_i\langle\Lambda\rangle_{ij} + A_{55}\langle\Lambda_{32}\rangle_{ij}$$

$$\bar{f}_{ijmn}^{\psi s} = A_{55}(\langle\Lambda_{33}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{33}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$\bar{f}_{ijmn}^{\psi f} = [\beta_n(\langle\Lambda\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda\rangle_{m+1,n}\delta_{i,m+1}) + A_{55}(\langle\Lambda_{34}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{34}\rangle_{m+1,n}\delta_{i,m+1})]\delta_{jn}$$

$$f_{ij}^{\psi w} = A_{55}\alpha_i\langle\Lambda\rangle_{ij} + \frac{A_{12}}{R}f_{ij}^{\psi t} + \frac{B_{12}}{R}(f_{ij}^{\psi h} - \alpha_i\langle\Lambda\rangle_{ij})$$

$$\bar{f}_{ij}^{\psi t} = f_{ij}^{\psi t},\quad \bar{f}_{ij}^{\psi h} = f_{ij}^{\psi h},\quad f_{ijmn}^{\psi s} = -\bar{f}_{ijmn}^{\psi s},\quad f_{ijmn}^{\psi f} = -\bar{f}_{ijmn}^{\psi f},\quad \bar{f}_{ij}^{\psi w} = f_{ij}^{\psi w}$$

$$\bar{f}_{ij}^{\phi t} = \left(A_{12} + \frac{B_{12}}{R}\right)\alpha_i\beta_j\langle\Lambda_{11}\rangle_{ij} + \left(A_{22} + \frac{B_{22}}{R}\right)\beta_j^2\langle\Lambda_{21}\rangle_{ij} + \left(B_{12} + \frac{D_{12}}{R}\right)\alpha_i\beta_j\langle\Lambda_{31}\rangle_{ij} + \left(B_{22} + \frac{D_{22}}{R}\right)\beta_j^2\langle\Lambda_{41}\rangle_{ij}$$

$$\bar{f}_{ij}^{\phi h} = \left(A_{12} + \frac{B_{12}}{R}\right)\alpha_i\beta_j\langle\Lambda_{12}\rangle_{ij} + \left(A_{22} + \frac{B_{22}}{R}\right)\beta_j^2\langle\Lambda_{22}\rangle_{ij} + \left(B_{12} + \frac{D_{12}}{R}\right)\alpha_i\beta_j\langle\Lambda_{32}\rangle_{ij} + \left(B_{22} + \frac{D_{22}}{R}\right)\beta_j^2\langle\Lambda_{42}\rangle_{ij}$$

$$\begin{aligned} f_{ijmn}^{\phi s} = & -(\alpha_{m-1}\langle\Lambda\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & -\left(A_{12} + \frac{B_{12}}{R}\right)\beta_n(\alpha_{m-1}\langle\Lambda_{13}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{13}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & -\left(A_{22} + \frac{B_{22}}{R}\right)\beta_n^2(\langle\Lambda_{23}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{23}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & -\left(B_{12} + \frac{D_{12}}{R}\right)\beta_n(\alpha_{m-1}\langle\Lambda_{33}\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda_{33}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \\ & -\left(B_{22} + \frac{D_{22}}{R}\right)\beta_n^2(\langle\Lambda_{43}\rangle_{m-1,n}\delta_{i,m-1} - \langle\Lambda_{43}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn} \end{aligned}$$

$$f_{ijmn}^{\phi s} = -\frac{1}{R}(\alpha_{m-1}\langle\Lambda\rangle_{m-1,n}\delta_{i,m-1} - \alpha_{m+1}\langle\Lambda\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$-\left(A_{12}+\frac{B_{12}}{R}\right)\beta_n(\alpha_{m-1}\langle\Lambda_{14}\rangle_{m-1,n}\delta_{i,m-1}-\alpha_{m+1}\langle\Lambda_{14}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$-\left(A_{22}+\frac{B_{22}}{R}\right)\beta_n^2(\langle\Lambda_{24}\rangle_{m-1,n}\delta_{i,m-1}-\langle\Lambda_{24}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$-\left(B_{12}+\frac{D_{12}}{R}\right)\beta_n(\alpha_{m-1}\langle\Lambda_{34}\rangle_{m-1,n}\delta_{i,m-1}-\alpha_{m+1}\langle\Lambda_{34}\rangle_{m+1,n}\delta_{i,m+1})\delta_{jn}$$

$$-\left(B_{22}+\frac{D_{22}}{R}\right)\beta_n^2(\langle\Lambda_{44}\rangle_{m-1,n}\delta_{i,m-1}-\langle\Lambda_{44}\rangle_{m+1,m}\delta_{i,m+1})\delta_{jn}$$

$$\bar{f}_{ij}^{\phi w}=-\frac{1}{R}\left(A_{22}+\frac{B_{22}}{R}\right)\beta_j\langle\Lambda\rangle_{ij}+\frac{A_{12}}{R}\bar{f}_{ij}^{\phi t}+\frac{B_{12}}{R}\bar{f}_{ij}^{\phi h}$$

$$f_{ij}^{\phi t}=-\bar{f}_{ij}^{\phi t},\quad f_{ij}^{\phi h}=-\bar{f}_{ij}^{\phi h},\quad \bar{f}_{ijmn}^{\phi s}=f_{ijmn}^{\phi s},\quad \bar{f}_{ijmn}^{\phi f}=f_{ijmn}^{\phi f},\quad f_{ij}^{\phi w}=-\bar{f}_{ij}^{\phi w}$$

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